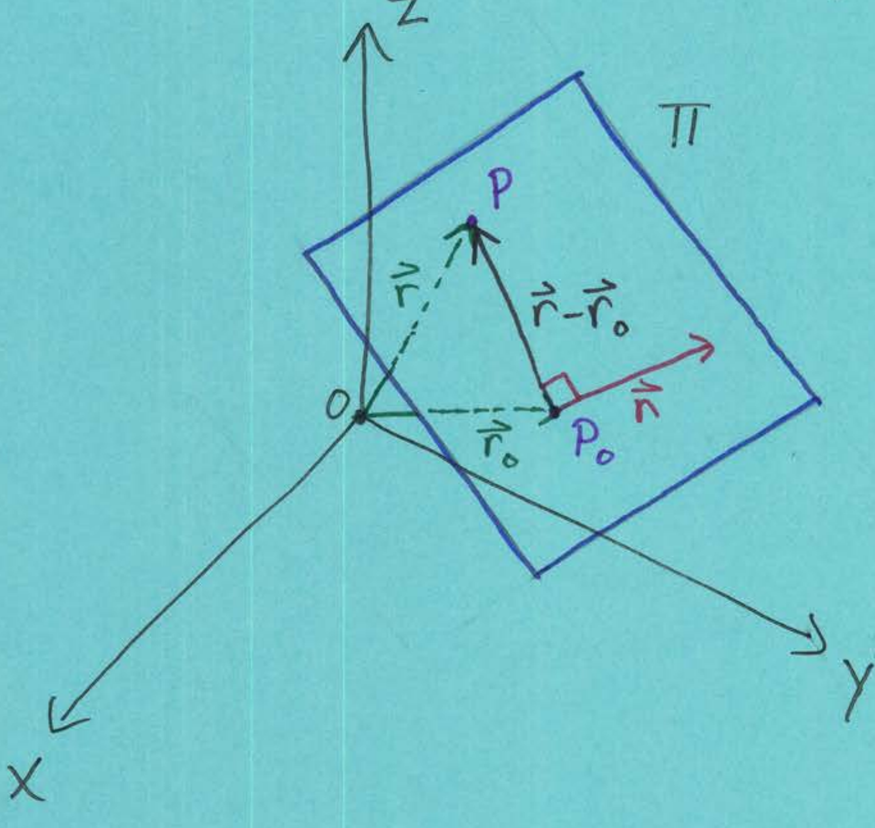
Lecture 6

Planes: Planes are determined by two pieces of information: 1) a point in the plane : $P_0 = (x_0, y_0, z_0)$

2) a vector normal (perpendicular) to the plane : $\vec{n}$



$P = (x, y, z)$ is any point in the plane

We know $\vec{n}$ is normal to the plane, and $\vec{r} - \vec{r}_0$ is a vector in the plane, so we must have:

$$\boxed{\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0}$$: vector equation of the plane Π.

In $\vec{n} = \langle a, b, c\rangle$, $\vec{r}_0 = \langle x_0, y_0, z_0\rangle$, $\vec{r} = \langle x, y, z\rangle$, the equation becomes:

$$\langle a,b,c\rangle \cdot (\langle x,y,z\rangle - \langle x_0,y_0,z_0\rangle) = \boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

This is called the <u>scalar equation of the plane</u> Π.

<u>Ex</u>: Find the equation of the plane passing through $(3,4,-3)$ with normal vector $\vec{n} = \langle 2,1,-1\rangle$

<u>Sol</u>: $\vec{r} - \vec{r}_0 = \langle x-3, y-4, z+3\rangle$, so

$$\vec{n}\cdot(\vec{r}-\vec{r}_0) = \langle 2,1,-1\rangle \cdot \langle x-3, y-4, z+3\rangle$$

$$= \boxed{2(x-3) + (y-4) - (z+3) = 0}$$

◇

We can also write the equation of a plane as a <u>linear equation</u> in x, y, and z:

$$\cancel{ax+by+cz=d} \qquad \boxed{ax+by+cz+d=0}$$

where $d = -(ax_0 + by_0 + cz_0)$.

Ex: Find an equation of the plane passing through $P=(0,1,1)$, $Q=(1,0,1)$, & $R=(1,1,0)$

Sol: We need a vector normal to the plane. We can accomplish this by taking the cross product of two vectors in the plane, namely: $\vec{PQ}$ & $\vec{PR}$

$$\vec{PQ}=\langle 1,-1,0\rangle \quad \& \quad \vec{PR}=\langle 1,0,-1\rangle$$

So,
$$\vec{n}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \langle 1-0, -(-1-0), 0-(-1)\rangle$$

$$\cancel{\langle 1,1,1\rangle} \qquad = \langle 1,1,1\rangle$$

So, an equation is:

$$1(x-0)+1(y-1)+1(z-1) = x+y-1+z-1=0$$

$$\Leftrightarrow x+y+z=2$$

◇

Ex: At what point does the line

$$x = 3+3t, \; y = t, \; z = -2+4t$$

intersect the plane $x+y+z=2$?

Sol: If the line intersects the plane, there has to be a point (x,y,z) on the line such that $x+y+z=2$. Since any point on L is described by $(x,y,z) = (3+3t, t, -2+4t)$, we plug this in. If we can solve for t, they intersect and we get the point of intersection. If not, they don't intersect. Plug in:

$$x+y+z = (3+3t) + (t) + (-2+4t) = 2$$

$$\Rightarrow 1+8t = 2 \Rightarrow 8t = 1 \Rightarrow t = 1/8$$

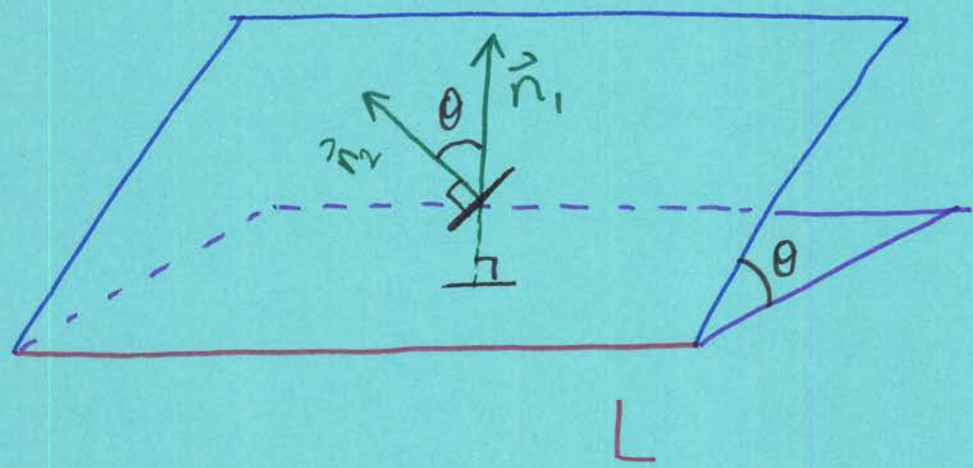
So, the point of intersection is:

$$(x,y,z) = \left(3+3\left(\tfrac{1}{8}\right), \tfrac{1}{8}, -2+4\left(\tfrac{1}{8}\right)\right) = \left(\tfrac{27}{8}, \tfrac{1}{8}, \tfrac{-3}{2}\right)$$

◇

What about plane intersections?

We say two <u>planes</u> are <u>parallel</u> if their normal vectors are parallel. If two planes are not parallel, then they intersect in a line.



(<u>Note</u>: A line L in a plane with normal vector $\vec{n}$ has direction vector $\perp \vec{n}$.)

<u>Example</u>: Do the planes $2x-3y+4z=5$ & $x+6y+4z=3$ intersect? If so, what is the angle of their intersection? also, give an expression for their line of intersection.

<u>Sol</u>: The planes have normal vectors

$$\vec{n}_1 = \langle 2,-3,4\rangle \ \& \ \vec{n}_2 = \langle 1,6,4\rangle$$

which we see are not parallel since $\vec{n}_1 \neq c\vec{n}_2$ for any c.

So the planes intersect. The angle of intersection is then the angle between their normal vectors. So if θ is the angle:

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{\langle 2,-3,4\rangle \cdot \langle 1,6,4\rangle}{\left(\sqrt{2^2+(-3)^2+4^2}\right)\left(\sqrt{1^2+6^2+4^2}\right)}$$

$$= \frac{2-18+16}{\sqrt{4+9+16}\,\sqrt{1+36+16}} = \frac{0}{\sqrt{29}\,\sqrt{53}} = 0$$

so $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ (Thus, the planes are, in fact, perpendicular.)

To find the line of intersection, we need a vector in the direction of the line. Let's call this vector $\vec{v}$. Since the line sits in both planes, we must have $\vec{v}\cdot\vec{n}_1 = 0$ & $\vec{v}\cdot\vec{n}_2 = 0$ since the normal vector of a plane is perpendicular to any vector in that plane. This means $\vec{v} \,//\, \vec{n}_1 \times \vec{n}_2$ ~~so we may as well choose~~

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 4 \\ 1 & 6 & 4 \end{vmatrix} = \langle -12-24, -(8-4), 12-(-3)\rangle$$
$$= \langle -36, -4, 15\rangle$$

we need a point on the line, i.e., a point in both planes. This is usually most easily done by choosing one of x, y, z to be some number and solving for the other two. Let's choose $z=0$ here. Then we must solve

$$\begin{cases} 2x-3y=5 & ① \\ x+6y=3 & ② \end{cases}$$

which we get by plugging in $z=0$ to the equation for each plane.

Solving ② for x gives: $x=3-6y$. Plug this in ①:

$$2(3-6y)-3y=6-15y=5 \Rightarrow -15y=-1 \Rightarrow y=\frac{1}{15}.$$

Then $x=3-6\left(\frac{1}{15}\right)=3-\frac{2}{5}=\frac{13}{5}$. So a point on the line of intersection is $\left(\frac{13}{5}, \frac{1}{15}, 0\right)$. So, the

- parametric equations for L are:

$$x=\frac{13}{5}-36t, \quad y=\frac{1}{15}-4t, \quad z=15t$$

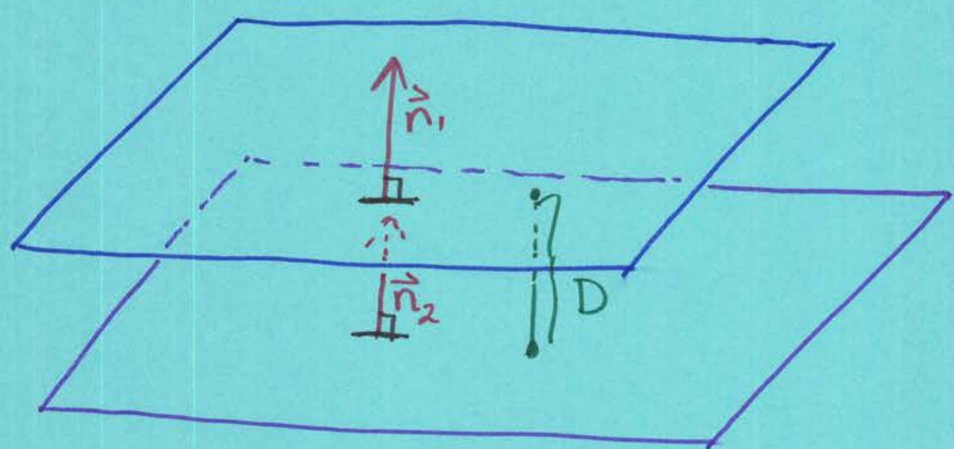
- symmetric equations for L are:

$$\frac{x-\frac{13}{5}}{-36}=\frac{y-\frac{1}{15}}{-4}=\frac{z}{15}$$

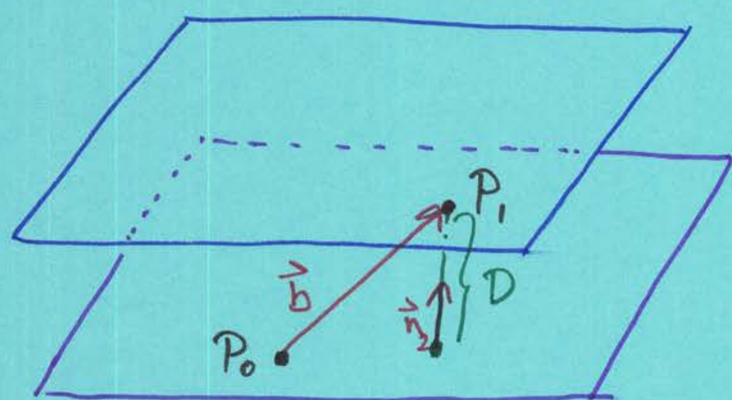
◇

Example: The planes $x-4y+2z=0$ and $2x-8y+4z=-1$ are parallel. Find the distance between them.

Sol: We have a situation like this:



How can we the distance D? Observe:



$\vec{b} = \overrightarrow{P_0P_1}$

Thus we can see,

$$D = \left|\text{proj}_{\vec{n}_2} \vec{b}\right| = \left|\text{comp}_{\vec{n}_2} \vec{b}\right| = \frac{|\vec{n}_2 \cdot \vec{b}|}{|\vec{n}_2|}$$

vector magnitude

absolute value of a number

So, we need a point on each plane:

$(0,0,0)$ is a point on $x-4y+2z=0$

$(-\frac{1}{2},0,0)$ is a point on $2x-8y+4z=-1$

It doesn't matter which point we take as P_0 or P_1, so choose $P_0=(0,0,0)$ & $P_1=(-\frac{1}{2},0,0)$.

It also doesn't matter which normal vector we take since they are parallel. Let's use $\vec{n}=\langle 1,-4,2\rangle$. Thus, the distance between the planes is:

$$D=\frac{|\vec{n}\cdot\overrightarrow{P_0P_1}|}{|\vec{n}|}=\frac{|\langle 1,-4,2\rangle\cdot\langle -\frac{1}{2},0,0\rangle|}{\sqrt{1^2+(-4)^2+2^2}}=\frac{|-\frac{1}{2}+0+0|}{\sqrt{21}}$$

$$=\frac{1/2}{\sqrt{21}}=\frac{1}{2\sqrt{21}}$$

◇

From our discussion, we obtained the useful fact:

The distance from a point $P_1 = (x_1, y_1, z_1)$ to the plane passing through the point $P_0 = (x_0, y_0, z_0)$ with normal vector $\vec{n}$ is:

$$D = \frac{|\vec{n} \cdot \vec{b}|}{|\vec{n}|}, \text{ where } \vec{b} = \overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

If we write the plane as, $ax + by + cz + d = 0$, we can show

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

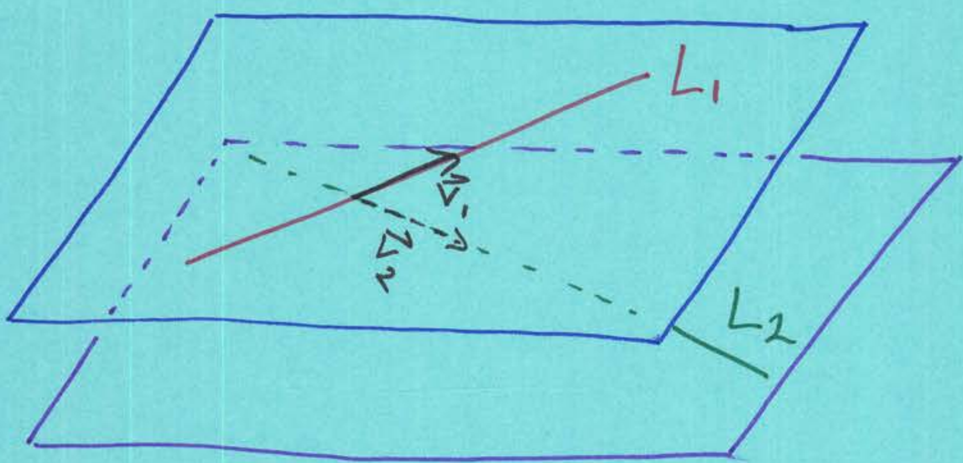
Ex: We showed on Wednesday that the lines

L_1: $x = 3 + 2t$, $y = 4 - t$, $z = 1 + 3t$

L_2: $x = 1 + 4s$, $y = 3 - 2s$, $z = 4 + 5s$

are skew. Find the distance between them.

Sol: Since the lines are skew, we can view them as lying in two parallel planes:



This question, then, is asking us to find the distance between these planes. We need a normal vector to these planes, and since this vector is perpendicular to L_1 & L_2, we can find it as $\vec{n} = \vec{v}_1 \times \vec{v}_2$ where $\vec{v}_1$ & $\vec{v}_2$ are direction vectors of L_1 & L_2, respectively. $\vec{v}_1 = \langle 2, -1, 3 \rangle$ & $\vec{v}_2 = \langle 4, -2, 5 \rangle$ are good choices. Then

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 4 & -2 & 5 \end{vmatrix} = \langle -5-(-6), -(10-12), -4-(-4) \rangle$$

$$= \langle 1, 2, 0 \rangle$$

A point on L_1 is $(3,4,1)$ $(t=0)$ and on L_2 is $(1,3,4)$ $(s=0)$. So the distance is:

$$D = \frac{|\vec{n} \cdot \vec{b}|}{|\vec{n}|} = \frac{|\langle 1,2,0 \rangle \cdot \langle 1-3, 3-4, 4-1 \rangle|}{\sqrt{1^2+2^2+0^2}} = \frac{|\langle 1,2,0 \rangle \cdot \langle -2,-1,3 \rangle|}{\sqrt{5}} = \frac{4}{\sqrt{5}}$$